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Universal critical behavior of the synchronization transition in delayed chaotic systems

Ivan G. Szendro[1,2,*] and Juan M. López[1,†]
[1] *Instituto de Física de Cantabria (IFCA), CSIC-UC, E-39005 Santander, Spain*
[2] *Departamento de Física Moderna, Universidad de Cantabria, Avenida. Los Castros, E-39005 Santander, Spain*
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We numerically investigate the critical behavior of the synchronization transition of two unidirectionally coupled delayed chaotic systems. We map the problem to a spatially extended system to show that the synchronization transition in delayed systems exhibits universal critical properties. We find that the synchronization transition is absorbing and generically belongs to the universality class of the bounded Kardar-Parisi-Zhang equation, as occurs in the case of extended systems. We also argue that directed percolation critical behavior may emerge for systems with strong nonlinearities.

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Synchronization of chaotic systems has attracted much interest in recent years and examples include chemical reactions, neuronal networks, Josephson junctions, electronic circuits, and semiconductor lasers, among others (see Ref. [1] and references therein). More recently, and from a practical point of view, this burst of activity in the field is partially due to the potential applications in control and secure communications. It is expected that an increased complexity of the attractor would make it much more difficult to extract the dynamical information. In particular, delayed dynamical systems have been suggested as the ideal candidates for secure communication for several reasons. On the one hand, they are hyperchaotic systems with an arbitrarily large number of positive Lyapunov exponents, whose number increases linearly with the delay time [2,3]. On the other hand, they may be experimentally realized in the form of fast communication optical systems by using different types of delayed feedback setups [4–10].

Synchronization of two separate delayed chaotic systems is achieved by allowing some communication between them. The possible schemes are diverse, including variable substitution, symmetric feedback, etc. Generally, there exists a critical coupling constant κ_c that separates two different phases. For low coupling values, $\kappa < \kappa_c$, there is a disordered phase in which each system evolves independently and the time-average difference between both systems remains finite. In contrast, a synchronized phase appears for $\kappa > \kappa_c$, in which the average error tends to zero and memory of the initial difference is asymptotically lost.

Very recent studies have been devoted to investigate important aspects of synchronization in delayed dynamical systems as, for instance, analytical approximations to estimate the synchronization threshold [11], the robustness of the transition to parameter mismatch [12], chaos control in lasers with feedback [13], information flow between drive and response systems [14], and the effect of a time-dependent delay [15]. However, the mechanism behind the synchronization transition in delayed dynamical systems and its relationship with those exhibited by other chaotic systems with many degrees of freedom is still unknown.

Remarkably, synchronization also takes place in coupled spatially extended systems with many degrees of freedom and space-time chaos. In this case, the synchronization transition has been shown to be an *absorbing* nonequilibrium phase transition and, accordingly, its critical properties have attracted much interest in the past few years [16–21]. Despite being scalar (i.e., described by only one dynamical variable), delayed dynamical systems are formally infinite dimensional dynamical systems and show many aspects of space-time chaos, including the formation and propagation of structures, defects, and spatiotemporal intermittency [22–25]. An interesting question that naturally arises is whether the synchronization transition in (scalar) hyperchaotic systems with delayed feedback could also be understood as a nonequilibrium phase transition, as occurs in (vectorial) extended dynamical systems with space-time chaos.

In this paper, we characterize the synchronization transition in unidirectionally coupled delayed dynamical systems as a nonequilibrium critical phase transition and relate it to existing universality classes. We exploit the interpretation of delayed dynamical systems as spatially extended systems [22–25] to show that the synchronization transition in delayed systems exhibits *universal* properties, which are independent of microscopic details of the individual systems being coupled. We find that the synchronization transition generically belongs to the universality class of the bounded Kardar-Parisi-Zhang (BKPZ) equation, as occurs in the case of extended systems [16–21]. We also argue that directed percolation (DP) critical behavior may emerge for systems with strong nonlinearities. Our results show that the critical properties of the synchronization transition in delayed chaotic systems are identical to those in spatially extended systems, despite being the former a scalar system with no real spatial structure.

We consider two identical time-delay systems described by two coupled differential-delay equations, the drive (transmitter) system

$$\dot{u} = -au + F(u_\tau), \tag{1}$$

and the response (receiver) system

*Electronic address: szendro@ifca.unican.es
†Electronic address: lopez@ifca.unican.es

$$\dot{v} = -av + F(v_\tau) + \kappa(u - v), \quad (2)$$

where $u_\tau = u(t-\tau)$ and $v_\tau = v(t-\tau)$ are the delayed variables, τ is the time delay, and κ is the coupling strength.

Delayed systems like Eq. (1) are used in a variety of applications ranging from biology to optics. We have studied in detail three prototypical models: the Mackey-Glass model [26], $F(\rho) = b\rho/(1+\rho^{10})$ (initially introduced to describe the regulation of blood cell production in patients with leukemia); the Ikeda differential-delay equation [4], $F(\rho) = b\sin(\rho - \rho_0)$ (which appears in the context of optical feedback on a laser beam [4] and experimental setups of optical generators of chaos in wavelength [8,9]); and a model with the piecewise linear delay expression given by $F(\rho) = 2\rho$ if $\rho \leqslant 1/2$, and $F(\rho) = 2 - 2\rho$ if $\rho > 1/2$. We have studied the synchronization critical properties by means of computer simulations of these three systems and found similar results.

In order to make apparent the existence of a nonequilibrium phase transition we transform the pair of coupled delayed systems Eqs. (1) and (2) into two coupled spatially extended systems. This can be readily done by introducing the coordinate transformation, $t = x + \theta\tau$, where $x \in [0, \tau]$ is the *space variable*, while $\theta \in \mathbb{N}$ is a discrete time variable [22]. Note that the time delay τ becomes the *system size* in such a way that the time dependence with the delayed variable is transformed into an interaction within the horizontal space coordinate x in the space-time representation. This is a powerful representation in which delayed systems can be treated as extended systems to identify many features of space-time chaos [22–24].

Complete synchronization of the two coupled delayed systems, Eqs. (1) and (2), is achieved if the synchronization error $u(t) - v(t)$ vanishes for all times t as $t \to \infty$. In the spatial picture we replace the dynamical variables $u(t)$ and $v(t)$ by $\tilde{u}(x,\theta)$ and $\tilde{v}(x,\theta)$, so that the synchronization error is given by $w(x,\theta) = \tilde{u}(x,\theta) - \tilde{v}(x,\theta)$, and synchronization occurs when $w(x,\theta)$ vanishes at all x for $\theta \to \infty$. This is equivalent to a vanishing spatial average $\langle |w(x,\theta)| \rangle_x$. Note that the spatial average in the space-time representation corresponds to the average within the delay time τ. In contrast, one has $\langle |w(x,\theta)| \rangle_x > 0$ in the unsynchronized state. This makes the average error $w(\theta) \equiv \langle |w(x,\theta)| \rangle_x$ a natural order parameter for the transition. Critical properties of the synchronization transition can now be studied by analyzing the dependence of the order parameter on the coupling strength κ. In addition, the analysis of the critical behavior for finite time delays can be naturally carried out by standard finite-size scaling techniques. The remaining part of this paper is devoted to the study of these issues.

Our findings are based upon extensive numerical simulations of the three time-delay systems introduced above. In all our numerical simulations we have used the Adams-Bashforth-Moulton predictor-corrector scheme [27] to integrate the coupled differential-delay equations (1) and (2). For the sake of brevity we focus the discussion of our numerical results on the Mackey-Glass model, but we found similar results for the Ikeda equation and the piecewise linear system. The parameters $a=1$ and $b=2$ are used in all the results

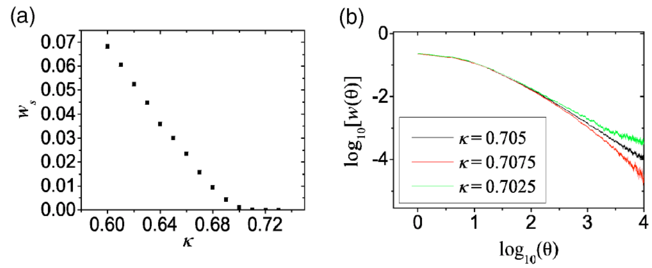

FIG. 1. (Color online) (a) The average synchronization error in the stationary state, w_s, is plotted near the synchronization threshold for $\tau = 200$. Each point corresponds to an average over 600 realizations. (b) The spatial average of the synchronization error, $w(\theta)$, for $\tau = 2000$ and three different values of κ, each curve being an average over 300 realizations is shown. $\kappa_c = 0.705$ is obtained and the corresponding slope yields $\delta = 1.14$.

we are presenting here, and simulations with time delays varying from ten to a few thousand time units have been carried out using an integration step of $\Delta t = 0.01$. The region of interest here corresponds to delays $\tau \gg 1.7$ for which the Mackey-Glass model is hyperchaotic [2].

In Fig. 1(a) we present our results for the order parameter, i.e., the average synchronization error, in the stationary state $w_s(\kappa) = \lim_{\theta\to\infty} \langle |w(x,\theta)| \rangle_{x,\theta}$ for a system of size (delay) $\tau = 200$ as the coupling strength is varied. Inspection of Fig. 1(a) indicates that the transition is continuous and occurs roughly around $\kappa = 0.7$, which is in agreement with earlier estimations for the same model parameters [11]. Dynamic critical behavior is studied by calculating the indices δ and β that describe the critical behavior of the order parameter near the threshold for synchronization: $w(\theta) \sim \theta^{-\delta}$ for $\kappa = \kappa_c$ and $w_s(\kappa) \sim |\kappa - \kappa_c|^\beta$ as the transition is approached for $\kappa < \kappa_c$. Studying critical behavior demands us to obtain a good estimation for the critical threshold which implies the use of large delays (system sizes). In Fig. 1(b) we plot the subcritical and supercritical behavior of the average synchronization error for a large system size $\tau = 2000$, as the transition is approached from below ($\kappa = 0.7025$) and above ($\kappa = 0.7075$), respectively. Within our numerical resolution we find that the best power-law behavior $w(\theta) \sim \theta^{-\delta}$ is obtained at $\kappa_c = 0.705 \pm 0.002$, which gives an estimation for the critical exponent $\delta = 1.14 \pm 0.03$.

Once the critical threshold has been determined, we use finite-size scaling at the critical point $\kappa_c = 0.705$ and fit numerical data to the scaling form

$$w(\theta) = \theta^{-\delta} f(\theta/\tau^z), \quad (3)$$

where the scaling function $f(y) \sim \text{const}$ for $y \ll 1$ and $f(y) \sim y^\delta$ for $y \gg 1$. This gives us an independent determination of δ and the dynamic exponent z. In Fig. 2(a) we show numerical results for different system sizes at the critical point κ_c, and these data are best collapsed in Fig. 2(b) with $z = 1.45 \pm 0.05$ and $\delta = 1.15 \pm 0.05$ (the latter in good agreement with our previous estimate in Fig. 1).

Next we report on off-critical numerical calculations of the synchronization error. This allows us to estimate the correlation length exponent. In Fig. 3(a) we plot the order pa-

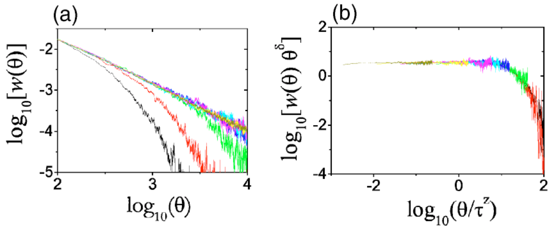

FIG. 2. (Color online) Finite-size data in (a) unscaled and (b) scaled coordinates for $\tau \in \{10, 20, 50, 100, 200, 150, 500, 2000\}$ and $\kappa_c = 0.705$. The collapse is obtained for $\delta = 1.15$ and $z = 1.45$. Every curve corresponds to an average over 60 realizations.

rameter as the coupling strength is varied close to the synchronization threshold for a large system size $\tau = 2000$. Near and below the transition the characteristic size of synchronized regions within the disordered phase is given by the horizontal correlation length ξ and is expected to diverge as $\xi \sim \epsilon^{-\nu_\perp}$ when the distance to the critical point tends to zero, $\epsilon = |\kappa - \kappa_c| \to 0$. Correspondingly, the characteristic time ϑ measuring the duration of a fluctuation of size ξ diverges as $\vartheta \sim \xi^z \sim \epsilon^{-\nu_\parallel}$, where $\nu_\parallel = \nu_\perp z$. Off-critical data are then expected to satisfy the scaling form $w(\theta, \epsilon) = \theta^{-\delta} g(\theta/\vartheta)$, so that numerical data in Fig. 3(a) must collapse according to

$$w(\theta, \epsilon) = \theta^{-\delta} g(\theta \epsilon^{\nu_\parallel}) \tag{4}$$

for the appropriate election of the critical exponents δ and $\nu_\parallel$. In Fig. 3(b) we show a data collapse for the exponents $\delta = 1.05 \pm 0.05$, and $\nu_\parallel = 1.4 \pm 0.1$, where the two branches correspond to numerical data for coupling strengths above and below critical. Also the index β can be obtained from the scaling behavior Eq. (4); for $\theta \gg \vartheta$ we have $w(\theta \to \infty, \epsilon) \sim \epsilon^{\beta}$, where $\beta = \nu_\parallel \delta = 1.47$.

The critical exponents of the synchronization transition in delayed systems are to be compared with those observed in extended systems. This will allow us to identify the mechanisms behind the transition in both types of high-dimensional dynamical systems. In the context of extended systems, the exponential growth rate of the error $|w(x,t)|$ is known as the transverse Lyapunov exponent $\lambda_\perp$ and measures the stability of the synchronized solution $w(x,t) = 0$. Accordingly, stable synchronization implies that the transverse Lyapunov exponent must be negative. For dynamical systems with *smooth* local nonlinearities (such as lattices of logistic or tent-coupled maps), this proves to be a sufficient condition as well. In this case, the synchronization transition is found to be generically in the universality class of the KPZ equation with a (bounding) growth-limiting term, the so-called BKPZ universality [17–21]. Numerical estimates of the critical exponents gave $\delta_{\mathrm{BKPZ}} = 1.17 \pm 0.05$, $\beta_{\mathrm{BKPZ}} = 1.50 \pm 0.05$, and $z_{\mathrm{BKPZ}} = 1.53 \pm 0.05$ for different models studied in the recent literature [18–21]. On the contrary, in the presence of *strong* and localized nonlinearities (such as, for instance, for Bernoulli-coupled maps), the synchronized phase turns out to be unstable even for negative values of $\lambda_\perp$ [18]. In this case, the transition occurs only when the propagation velocity of finite-amplitude perturbations vanishes. The critical properties of the transition are then associated with the DP universality class [18,21]. The fraction of nonsynchronized sites corresponds to the fraction of active sites in DP. Correspondingly, the critical exponents are given by $\delta_{\mathrm{DP}} = 0.159$, $\beta_{\mathrm{DP}} = 0.277$, $z_{\mathrm{DP}} = 1.581$ [18,21]. The DP correlation length and time exponents are known to be $\nu_\perp = 1.10$ and $\nu_\parallel = 1.73$, respectively [28].

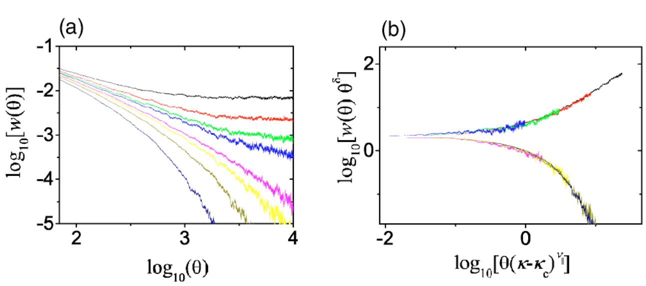

FIG. 3. (Color online) Off-critical data in (a) unscaled and (b) scaled coordinates for $\tau = 2000$ and various values of $\kappa \in (0.685, 0.725)$. The collapse is obtained for $\kappa_c = 0.705$, $\delta = 1.05$, and $\nu_\parallel = 1.4$. Every curve corresponds to an average over 60 realizations.

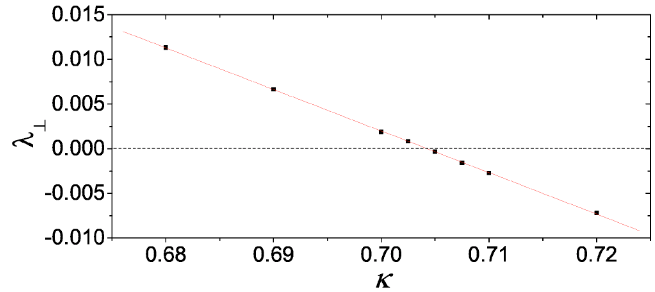

FIG. 4. (Color online) The transverse Lyapunov exponent, $\lambda_\perp$, is plotted for $\tau = 2000$ and various values of κ around the synchronization threshold. Note that $\lambda_\perp$ changes sign near κ_c.

Our numerical results, $\delta = 1.15$, $\beta = 1.47$, $z = 1.45$, and $\nu_\parallel = 1.14$, strongly suggest that the synchronization transition in delayed chaotic systems generically belongs to the BKPZ universality class, as occurs in extended chaotic systems. As an additional check, we have measured the transverse Lyapunov exponent $\lambda_\perp$ for the coupled delayed systems, Eqs. (1) and (2), with different coupling strengths and, as shown in Fig. 4, we found that the transition takes place when $\lambda_\perp$ becomes negative, as expected for the BKPZ universality class. Nevertheless, the nature of the transition can be changed to DP behavior in the presence of strong local nonlinearities akin to what occurs in extended systems. In fact, by choosing the nonlinear function $F(\rho) = 2\rho \bmod 1$ the exponent δ drops to $\delta = 0.16 \pm 0.03$ in good agreement with DP. The different nature of the transitions in the two cases can be seen in Fig. 5, where we show the spatiotemporal evolution of the synchronization error $|w(x,\theta)|$ for the (a) smooth Mackey-Glass and (b) strongly nonlinear models, respectively, just slightly above the transition.

In conclusion, we have studied the critical properties of the synchronization transition in unidirectionally coupled delayed chaotic systems. We used a standard coordinate transformation to map the (scalar) time-delay system to a spatially

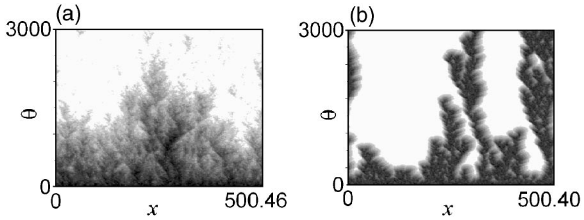

FIG. 5. The synchronization error for a coupling slightly above the synchronization threshold, $\kappa \gtrsim \kappa_c$, is plotted for $\tau = 500$ in the cases of (a) smooth nonlinearities and (b) strong local nonlinearities. The horizontal width has been chosen slightly larger than τ to eliminate the systematic drift. Compare with Fig. 2 of Ref. [18].

extended system. This mapping allowed us to study the synchronization transition as an absorbing nonequilibrium phase transition. Comparison of the critical exponents as well as the behavior of the transverse Lyapunov exponent lead us to conclude that the synchronization transition in delayed systems generically belongs to the BKPZ universality class independently of the specific form of the delay expression, as long as it is a continuous function, just as occurs for synchronization of space-time chaos. Finally, our numerical results also indicate that the existence of discontinuities in the delay nonlinear function may change this critical behavior from BKPZ to DP, which suggests that the same mechanisms that produce DP behavior in coupled extended systems can be invoked in the case of delayed chaotic systems.

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